

A Dynamical System Analysis of $f(R, T)$ Gravity

Behrouz Mirza* and Fatemeh Oboudiat†

Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

We investigate equations of motion and future singularities of $f(R, T)$ gravity where R is the Ricci scalar and T is the trace of stress-energy tensor. Future singularities for two kinds of equation of state (barotropic perfect fluid and generalized form of equation of state) are studied. While no future singularity is found for the first case, some kind of singularity is found to be possible for the second. We also investigate $f(R, T)$ gravity by the method of dynamical systems and obtain some fixed points. Finally, the effect of the Noether symmetry on $f(R, T)$ is studied and the consistent form of $f(R, T)$ function is found using the symmetry and the conserved charge.

I. INTRODUCTION

Recent cosmological observations show that our universe has an accelerating expansion [1]. Two groups of solutions are available that can be invoked for explaining the phenomenon. The first one is based on the belief that some exotic matter exist within the framework of General Relativity (GR) known as dark energy that has the parameter $\omega < 0$ in its equation of motion. Such a matter raises some fundamental questions such as the existence of negative entropy, future singularities, and the violation of some energy conditions.

On the other hand, some authors have generalized GR to some new theories of gravity [2]. Finite time future singularities of these theories was studied in [3]. One of these is $f(R)$ gravity in which the standard Einstein-Hilbert action is replaced with an arbitrary function of Ricci scalar [4]. In addition to its capability to describe the expansion of the universe without introducing any dark energy [5], this generalized theory of gravity has other advantages. For example, it can explain the dynamics of galaxies without recourse to the concept of dark matter [6] and unifies inflation with dark energy [7–11].

A further generalization of $f(R)$ is $f(R, T)$ gravity where T is the trace of stress-energy tensor [12]. As a consequence of using stress-energy tensor as a source, the motion of the particles does not take place along a geodesic path because there is an extra force perpendicular to the four-velocity unless we add the constraint of conservation of stress-energy tensor (unlike GR and $f(R)$ theories, the continuity equation is independent of equations of motion in this case). It is shown in [13] that due to the conservation of stress-energy tensor, T sector of $f(R, T)$ cannot be chosen arbitrarily but it has a special form. The thermodynamics of this model is studied in [14], and the possibility of wormhole geometry is examined in [15]. Also energy conditions [16], cosmological solutions [17], scalar perturbations [13] are investigated. In [18] solar system consequences of the model is argued. Further generalization of this theory to $f(R, T, R_{\mu\nu}T^{\mu\nu})$ is proposed in [19].

The method of autonomous dynamical systems is a use-

ful tool for investigating the modified theories of gravity such as $f(R)$ and some other theories [20–23]. In [24] the method is investigated for $f(R, T)$ theory assuming conservation of energy. In this paper we study the method by no limiting condition on energy.

The concept of symmetry has always been an attractive subject in physics. Noether symmetry attracts more attention because it helps to find constants of motion (like energy and momentum) from continuous symmetries of the system. Some efforts has been done to look for such conserved quantities in cosmological models [25]. Some authors have considered the effect of Noether symmetry in extended theories of gravity such as $f(R)$ [26] and $f(T)$ theories [27]. We investigate the effect of Noether symmetry on $f(R, T)$ theory to see if it is possible to make a consistent form of $f(R, T)$ by Noether symmetry.

This article is organized as follows. In section 2, we briefly explain the action of the $f(R, T)$ model and obtain the gravitational field equations. In section 3, we consider singularities for dark energy. The aim of section 4 is to consider the method of dynamical systems in $f(R, T)$ theory. In section 5 the Noether symmetry is studied. Section 6 concludes with a summary and discussion.

II. EQUATIONS OF MOTION

General form of the action for $f(R, T)$ model is:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2k} f(R, T) + \mathcal{L}_m \right\} \quad (1)$$

where $k = 8\pi G$, G is Newtonian constant, R is Ricci scalar, T is the trace of stress-energy tensor and \mathcal{L}_m is matter lagrangian density. By varying the action with respect to the metric we can find equations of motion:

$$\begin{aligned} f_R R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) f_R \\ = k T_{\mu\nu} - f_T (T_{\mu\nu} + \Theta_{\mu\nu}) \end{aligned} \quad (2)$$

where:

$$\begin{aligned} \Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g_{\mu\nu}} = -2T_{\mu\nu} + g_{\mu\nu} \mathcal{L}_m - 2g^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}} \\ f_R = \frac{\partial f(R, T)}{\partial R} \quad f_T = \frac{\partial f(R, T)}{\partial T} \end{aligned} \quad (3)$$

* b.mirza@cc.iut.ac.ir

† f.oboudiat@ph.iut.ac.ir

The following equations can also be obtained easily

$$\nabla^\mu G_{\mu\nu} = 0 \quad (4)$$

$$\nabla^\mu (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\alpha \nabla^\alpha) f(R, T) = R_{\mu\nu} \nabla^\mu f(R, T) \quad (5)$$

where $G_{\mu\nu}$ is the Einstein tensor. By contracting (2) into ∇^μ and using (4) and (5) we have:

$$\begin{aligned} (T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu f_T + f_T \nabla^\mu \Theta_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\nabla^\mu T) f_T \\ = (k - f_T) \nabla^\mu T_{\mu\nu} = j_\nu \end{aligned} \quad (6)$$

Replacing $f(R, T)$ by $f(R)$ in equation (6) leads to the conservation of energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$. This means that symmetries of $f(R)$ theory implies conservation of the energy momentum tensor. However, in $f(R, T)$ theory the energy momentum tensor is not generally conserved. In this and the next section we study $f(R, T)$ by assuming the conservation of energy and in the following sections we investigate more general cases. Assuming $\nabla^\mu T_{\mu\nu} = 0$ in equation (6) we obtain the following equation,

$$(T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu f_T + f_T \nabla^\mu \Theta_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\nabla^\mu T) f_T = 0 \quad (7)$$

Now, we concentrate on the case $f(R, T) = R + g(T)$. Assuming the matter content of the universe as a perfect fluid ($T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$) and having a FRW universe, i.e.:

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + d\Omega^2 \right) \quad (8)$$

the Friedman equations become:

$$3H^2 + \frac{\kappa}{a^2} = [k + g'(T)] \rho + g'(T)p + \frac{1}{2}g(T) \quad (9)$$

$$-2\dot{H} - 3H^2 - \frac{\kappa}{a^2} = kp - \frac{1}{2}g(T) \quad (10)$$

The continuity equation is independent from the Friedman equations:

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (11)$$

Adding the equation of state $p = p(\rho)$ to the three equations (9), (10) and (11), we have four independent equations which can be used to obtain the time dependence of four parameters of ρ, p, a , and $g(T)$. To do this for flat universe ($\kappa = 0$) we add the Friedman equations (9) and (10):

$$-2\dot{H} = [k + g'(T)](\rho + p) \quad (12)$$

Eliminating the term $\rho + p$ from (11) and (12), we have:

$$6H\dot{H} = [k + g'(T)]\dot{\rho} \quad (13)$$

By differentiating (9) and substituting (13), we have:

$$\dot{g}'(T)(\rho + p) + \dot{\rho}g'(T) + \frac{1}{2}\dot{g}(T) = 0 \quad (14)$$

By solving this equation for a perfect barotropic fluid ($p = \omega\rho$), we have for $\omega \neq \pm \frac{1}{3}, -1$:

$$g(T) = g_0 T^\theta, \quad \theta = \frac{1 + 3\omega}{2(1 + \omega)} \quad (15)$$

Also we found equation (15) for special case $f(R, T) = R + g(T)$ but it is possible to find the same relation for general case $f(R, T) = h(R) + g(T)$ from equation (6). The Friedman equations become:

$$3H^2 = k\rho + g_0(1 - 3\omega)^{\alpha-1} \rho^\alpha = k(\rho + \rho_{DE}) \quad (16)$$

$$2\dot{H} + 3H^2 = -kp + \frac{1}{2}g_0(1 - 3\omega)^\alpha \rho^\alpha = -k(p + p_{DE}) \quad (17)$$

We can interpret the above equations as the sum of matter fluid and DE in the framework of GR where density and pressure of DE are given by:

$$\rho_{DE} = g_0(1 - 3\omega)^{\alpha-1} \rho^\alpha \quad (18)$$

$$p_{DE} = -\frac{1}{2}g_0(1 - 3\omega)^\alpha \rho^\alpha = -\frac{1}{2}(1 - 3\omega)\rho_{DE} \quad (19)$$

Both fluids are, therefore, perfect with the equation of state parameter ω and $\omega_{DE} = -\frac{1}{2}(1 - 3\omega)$ [30]. To investigate the future singularities of this model, we first solve the continuity equation (11) as follows:

$$\rho = \rho_0 a^{-3(1+\omega)} \quad (20)$$

By substituting this equation in Friedman equation (9), we have:

$$\pm(t - t_0) = \int \frac{a^{\frac{1+3\omega}{2}} da}{\sqrt{d_1 + d_2 a^{\frac{3(1-\omega)}{2}}}} \quad (21)$$

where, $d_1 = \frac{k\rho_0}{3}$ and $d_2 = \frac{g_0\rho_0^\alpha(1-3\omega)^{\alpha-1}}{3}$. By substituting different allowed values of ω ($\omega \neq \pm \frac{1}{3}, -1$) in the above equation, we find no future singularities in this model. Below, we will consider the Friedman equations and future singularities for a more general equation of state.

We can interpret the Friedman equations (9) and (10) in a different way. If we define ρ_{de} and p_{de} and \tilde{k} as:

$$\rho_{de} = -p_{de} \equiv \frac{pg'(T) + \frac{1}{2}g(T)}{\tilde{k}} \quad (22)$$

$$\tilde{k} \equiv k + g'(T) \equiv 8\pi\tilde{G} \quad (23)$$

then, the Friedman equations become:

$$3H^2 = \tilde{k}(\rho + \rho_{de}) \quad (24)$$

$$-3H^2 - 2\dot{H} = \tilde{k}(p + p_{de}) \quad (25)$$

From above discussions it is obvious that there are some special features that we can distinguish this model from any typical dark energy model. One of the important features is that $f(R, T)$ does not preserve conservation of energy momentum tensor and continuity equation but it is possible to make the model to preserve it. The coupling constant can be constant (equations (9) and (10)) or have running with energy (equations (24) and (25)) like field theories QED, QCD, ... furthermore ρ_{de} and p_{de} have the behavior of dark energy so we can explain expanding of the universe without introducing any exotic matter like dark energy.

III. GENERALIZED EQUATION OF STATE

In this section, we consider a more general equation of state:

$$p = -\rho - f(\rho) \quad (26)$$

This kind of equation of state leads to five types of singularities in $f(R)$ theory [28, 29]:

- Type I ("Big Rip"): $t \rightarrow t_s, a \rightarrow \infty, \rho \rightarrow \infty$, and $|p| \rightarrow \infty$
- Type II ("Sudden"): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \rho_s$, and $|p| \rightarrow \infty$
- Type III: $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \infty$, and $|p| \rightarrow \infty$
- Type IV: $t \rightarrow t_s, a \rightarrow \infty, \rho \rightarrow 0$, and $|p| \rightarrow 0$, but higher derivatives of H diverges.
- Type V: In this type of singularity, $\omega = \frac{p}{\rho}$ diverges and it is possible that none of the other parameters has a singularity.

In the following, we will try to see if similar types of singularity exist in the $f(R, T)$ model. Assuming (26) and from the continuity equation (11), we have:

$$\dot{\rho} = 3Hf(\rho) \quad (27)$$

and from (14), we have:

$$g'(T) = g'_0 \sqrt{f(\rho)} a^3 \quad (28)$$

Eliminating $g'(T)$ between (12) and (28) we have:

$$2\dot{H} = \left(k + g'_0 \sqrt{f(\rho)} a^3 \right) f(\rho) \quad (29)$$

Having $a(t)$, we can now get the behavior of $\rho(t), p(t), g(t)$ and $f(\rho(t))$ from (26),(27),(28) and (29). If we assume $H(t)$ to have a singular form as:

$$H(t) = h(t_s - t)^{-m} \quad (30)$$

where t_s is the time of future singularity then for $m = 1$, $a(t)$ becomes:

$$a(t) = a_0(t_s - t)^{-h}. \quad (31)$$

The behavior of other functions near t_s will be as follows:

$$\rho, p \propto (t_s - t)^\alpha \quad (32)$$

$$g \propto (t_s - t)^\beta \quad (33)$$

$$g' \propto (t_s - t)^\gamma \quad (34)$$

The values of α, β and γ for different values of h are shown in Table I.

For $h > 0$, g and g' have singular behaviors near t_s and β and γ are their exponents but ρ and p are finite and their exponents are α . For $h < 0$, ρ and p are singular

TABLE I. The values of exponents of ρ, p, g , and g' for different values of h

h	α	β	γ
3	$\frac{14}{3}$	$-\frac{20}{3}$	$-\frac{5}{3}$
2	$\frac{8}{3}$	$-\frac{14}{3}$	$-\frac{11}{3}$
1	$\frac{2}{3}$	$-\frac{8}{3}$	$-\frac{17}{3}$
-1	-6	0	8
-2	-12	0	17
-3	-18	0	26

with the exponent α while g and g' are finite at t_s . The following relations holds between the exponents:

$$h > 0 : \beta_h = \frac{\alpha_h}{2} - 3h, \quad \alpha_h = -\beta_{h-1}, \quad \gamma_h = \beta_h - 1 \quad (35)$$

$$h < 0 : \beta_h = 0, \quad \alpha_h = 6h, \quad \gamma_h = 3h - 2 \quad (36)$$

For $m \neq 1$, $a(t)$ has the following different form:

$$a(t) = a_0 \exp \left[\frac{h(t_s - t)^{1-m}}{m-1} \right] \quad (37)$$

Depending on m , we have different results near t_s :

1. $m < -1$: It is obvious from the form of a, H, \dot{H} that none of them are singular near t_s and it is also clear from (27),(28) and (29) that $f(\rho), g'(T), \dot{\rho}$ are finite, too. In this case, we have no singularity in t_s except for the higher derivatives of H .
2. $-1 < m < 0$: In this case, at t_s , a and H are finite and just \dot{H} is singular, it is obvious from (27),(28) and (29) $f(\rho), g'(T), \dot{\rho}$ are singular. We can determine the behavior of ρ, p and g by numerical solution of equations (26),(27),(28) and (29).
3. $0 < m < 1$: In this case, H and \dot{H} are singular but a is finite. From (28) and (29) f and g' must be singular. Again, we see that all the parameters have singular behavior near t_s but p is negative in this case.
4. $m > 1$: In this case, all the three a, H, \dot{H} are singular. In this case, just g and g' are singular near t_s . ρ and p tend to zero from below.

We have summarized the above discussion about singularity in Table II. So we have following types of singularity in $f(R, T)$ model:

- Type \tilde{I} : $t \rightarrow t_s; H \rightarrow \infty, g \rightarrow \infty, \rho \rightarrow \infty, p \rightarrow \infty, g' \rightarrow \infty$; (for $0 < m < 1$).
- Type \tilde{II} : $t \rightarrow t_s; g \rightarrow \infty, \rho \rightarrow \infty, p \rightarrow \infty, g' \rightarrow \infty$; (for $-1 < m < 0$).

TABLE II. Singularity of cosmological parameters for the Hubble parameter $H(t) = h(t_s - t)^{-m}$

values of m	values of scale factor	values of other parameters
$m < -1$	a, H and \dot{H} are finite.	ρ, p, g and g' are finite.
$-1 < m < 0$	\dot{H} is singular.	ρ, p, g and g' are singular.
$0 < m < 1$	H and \dot{H} are singular.	ρ, p, g and g' are singular.
$m = 1$ $\begin{matrix} h > 0 \\ h < 0 \end{matrix}$	$\begin{matrix} a, H \text{ and } \dot{H} \text{ are singular.} \\ H \text{ and } \dot{H} \text{ are singular.} \end{matrix}$	$\begin{matrix} g \text{ and } g' \text{ are singular, } \rho \text{ and } p \text{ are finite.} \\ g \text{ and } g' \text{ are finite, } \rho \text{ and } p \text{ are singular.} \end{matrix}$
$m > 1$	a, H and \dot{H} are singular.	g and g' are singular, ρ and p are finite.

- Type \widetilde{III} : $t \rightarrow t_s; H \rightarrow \infty, g \rightarrow \infty, g' \rightarrow \infty; \rho, p$ finite (for $m = 1, h > 0$ and $m > 1$).
- Type \widetilde{IV} : $t \rightarrow t_s; H \rightarrow \infty, \rho \rightarrow \infty, p \rightarrow \infty; g, g'$ finite (for $m = 1, h < 0$).

It should be noted that the behaviors of g and g' exhibit the new characteristic of these types of singularities.

IV. $f(R, T)$ AND FIXED POINTS

Calculations of the previous section was a little restrictive. A special form of scale factor (31) was chosen in order to study future singularities of the theory and conservation of energy was imposed by hand. In this section we will consider a more general scenario. No special form of scale factor is chosen, the continuity equations are generalized to the following form:

$$\begin{aligned} \dot{\rho}_m + 3H(\rho_m + p_m) &= Q \\ \dot{\rho}_r + 3H(\rho_r + p_r) &= Q' \\ \dot{\rho}_T + 3H(\rho_T + p_T) &= -Q - Q' \end{aligned} \quad (38)$$

where, the indices m and r means matter and radiation, $\rho_T = \frac{1}{k} [g'(T)\rho_m + \frac{1}{2}g(T)]$ and p_T is equal to $-\frac{g(T)}{2k}$. It should be noted that T is the trace of energy momentum tensor of both matter and radiation; since energy momentum tensor of radiation is traceless ($p_r = \frac{1}{3}\rho_r$) T equals to the trace of energy momentum tensor of matter only i.e. $T = T_m$. Equation (38) leads the exchange of energy between matter, radiation and dark energy. Dimension of Q and Q' is like $\dot{\rho}$ or $H\rho$. We can put $H\rho_m, H\rho_r$, and $H\rho_T$ or a combination of them instead of Q and Q' . We choose a simple case $Q = \alpha H\rho_m$ and $Q' = \alpha H\rho_r$. So the

continuity equations can be written as follows:

$$\begin{aligned} \dot{\rho}_m + 3H(\rho_m + p_m) &= \alpha H\rho_m \\ \dot{\rho}_r + 3H(\rho_r + p_r) &= \alpha H\rho_r \\ \dot{\rho}_T + 3H(\rho_T + p_T) &= -\alpha(H\rho_m + H\rho_r) \end{aligned} \quad (39)$$

where α is a constant. The Friedman equations containing radiation and dust matter ($\omega_m = 0$) are as follows:

$$3H^2 = [k + g'(T)]\rho_m + \frac{1}{2}g(T) + k\rho_r \quad (40)$$

$$-2\dot{H} - 3H^2 = -\frac{1}{2}g(T) + \frac{k}{3}\rho_r \quad (41)$$

where indices r and m stand for radiation and matter. Due to complicated form of the above equations we use the method of autonomous dynamical system to study the problem. By introducing dimensionless parameters:

$$\begin{aligned} \Omega_m &= \frac{k\rho_m}{3H^2} \\ \Omega_r &= \frac{k\rho_r}{3H^2} \\ X &= \frac{g}{6H^2} \end{aligned} \quad (42)$$

and using modified Friedman equations (40) and (41) and continuity equations (39) the autonomous system becomes:

$$\begin{aligned} \frac{d\Omega_m}{dN} &= \Omega_m (\alpha + \Omega_r - 3X) \\ \frac{d\Omega_r}{dN} &= \Omega_r (\alpha - 1 + \Omega_r - 3X) \\ \frac{dX}{dN} &= \frac{\alpha - 3}{2}(1 - \Omega_m - \Omega_r - X) + 3X(1 + \frac{\Omega_r}{3} - X) \end{aligned} \quad (43)$$

where $N = \ln a$. The fixed points and the related physical parameters for the system (43) are represented in table III. Deceleration parameter related in the table is defined as bellow:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} = \frac{3}{2} \left(\frac{\Omega_r}{3} - X \right) + \frac{1}{2} \quad (44)$$

Looking for an accelerating universe we should find minus values for nowadays phase of the universe. The theory has four fixed points. To study the stability of the fixed points one has to study the eigenvalues of the first order perturbation matrix near the critical points which are presented in Table III by λ_i . The stable and unstable regimes are presented too.

A good cosmological model should contain at least a part of the standard cosmological model which is summarized as bellow [20]:

inflation \rightarrow radiation \rightarrow matter \rightarrow accelerating expansion.

Hence any matter and radiation fixed point in the model should be a saddle point and any accelerated phase should be an attractor.

The fixed point P_1 is hyperbolic (a hyperbolic fixed point is the one whose Jacobian matrix has no zero eigenvalue.) except at the values $\alpha = \pm 3, 4$. It represents dark energy dominated era ($\Omega_m = \Omega_r = 0$) and accelerated phase ($q < 0$) of the universe and it is stable at the values $-3 < \alpha < 3$.

P_2 is dark energy dominated and hyperbolic except at the values $1, \frac{5}{3}, -3$ and represents accelerated phase of the universe at the domain $\alpha < 1$. It is a stable fixed point in the regime $\alpha < -3$ and unstable for $\alpha > \frac{5}{3}$.

P_3 is a combination of radiation and dark energy. It can describe accelerated phase of the universe for $\alpha > 2$. There is no stable regime for P_3 while it is unstable at $\alpha < \frac{5}{3}$.

The only point that have dark energy and matter contribution is P_4 . It represents accelerated phase of the universe for $\alpha > 1$ and it is stable for $\alpha > 3$. The values of Ω_m and deceleration parameter are both proportional to $1 - \alpha$. This means near P_4 we have acceleration phase or $\rho_m < 0$ that is violation of weak energy condition. Evolution of the universe depends strongly on the values of the coupling parameter α . Stable and unstable fixed points for different regimes of α are summarized in table IV. In each regime the universe starts its evolution from unstable fixed point (as Big Bang) and ends in the stable one. Fig. 1 represents trajectories of the phase space for different regimes of table IV. As we see from table IV all stable fixed points are in an acceleration phase in the specified values of α and this is one of the successions of the theory. Matter and radiation dominated fixed points should be saddle points. The only regime that matter and radiation fixed points are saddle is $\frac{5}{3} < \alpha < 3$. In the regime $\frac{5}{3} < \alpha < \frac{10}{3}$, Ω_r becomes negative and weak energy condition is violated so the preferred regime is $\frac{10}{3} < \alpha < 3$. For these values of α , the saddle point P_3 is radiation dominated and there are some trajectories in Fig. 1e that evolves directly from P_4 to the final fixed point P_1 . This means transition of matter era to the acceleration phase.

In last Section by solving Eq. (39) for ρ_m and ρ_r and plugging them in the Friedman equations (40) and (41) a specific form of the $g(T)$ function was obtained. Although we generalized our calculations in this section but there are still some restrictions on the form of the conti-

nuity equation and also the form of the $f(R, T)$ function. In the next section we look at some aspects of symmetry in the $f(R, T)$ gravity to obtain a more complete view of it.

V. $f(R, T)$ AND NOETHER SYMMETRY

One of the tools to find symmetries of the theory is Noether symmetry approach. By the Noether theorem we know that there is a conserved charge related to every continuous symmetry of the theory that could be used to find the cyclic variables and reduce the dynamics of the system. Let \mathcal{L} be a lagrangian defined on tangent space $\mathcal{TQ} = \{q_i, \dot{q}_i\}$. A vector field on the tangent space can be represented by:

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial}{\partial \dot{q}^i} \quad (45)$$

where dot means derivative with respect to time and α^i are Noether functions. Lie derivative of lagrangian \mathcal{L} in the direction of X is defined as:

$$L_X \mathcal{L} = X \mathcal{L} = \alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \quad (46)$$

The condition:

$$L_X \mathcal{L} = 0 \quad (47)$$

implies that \mathcal{L} is conserved along the direction of X or X is a symmetry of \mathcal{L} . We have Euler-Lagrange equations too:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0 \quad (48)$$

contracting the above equation with α^i we have:

$$\alpha^i \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} \right) = 0 \Rightarrow \frac{d}{dt} \left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L} \quad (49)$$

We see from above equation that if X is a symmetry of \mathcal{L} (i.e. $L_X \mathcal{L} = 0$) the function:

$$\mathcal{A} = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \quad (50)$$

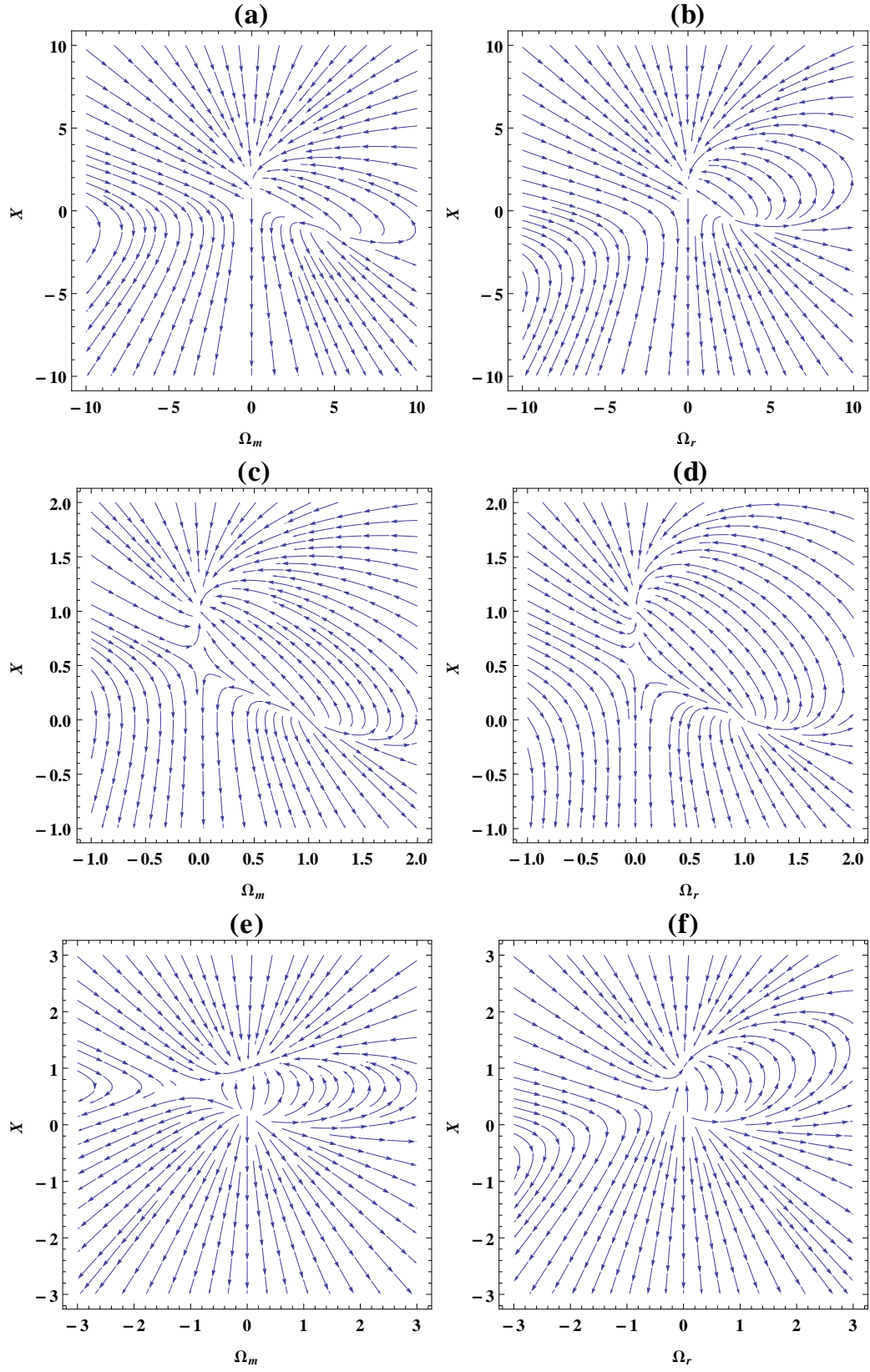
is the constant of the motion or conserved charge. This is the Noether theorem.

Now we want to find the consistent form of $f(R, T)$ by Noether symmetry and its conserved charge. General action of $f(R, T)$ theory is:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2k} f(R, T) + \mathcal{L}_m \right\} \quad (51)$$

From above discussion about Noether symmetry it is obvious that we need point like lagrangian to impose Noether constraint on the theory. To this aim we use Lagrange multipliers to set R and T as constraints of the motion. Integrating (51) we have:

$$S = 2\pi^2 \int dt \frac{a^3}{2k} \left\{ f(R, T) - \lambda \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) \right] - \lambda' [T - (\rho - 3p)] + 2k \mathcal{L}_m \right\} \quad (52)$$



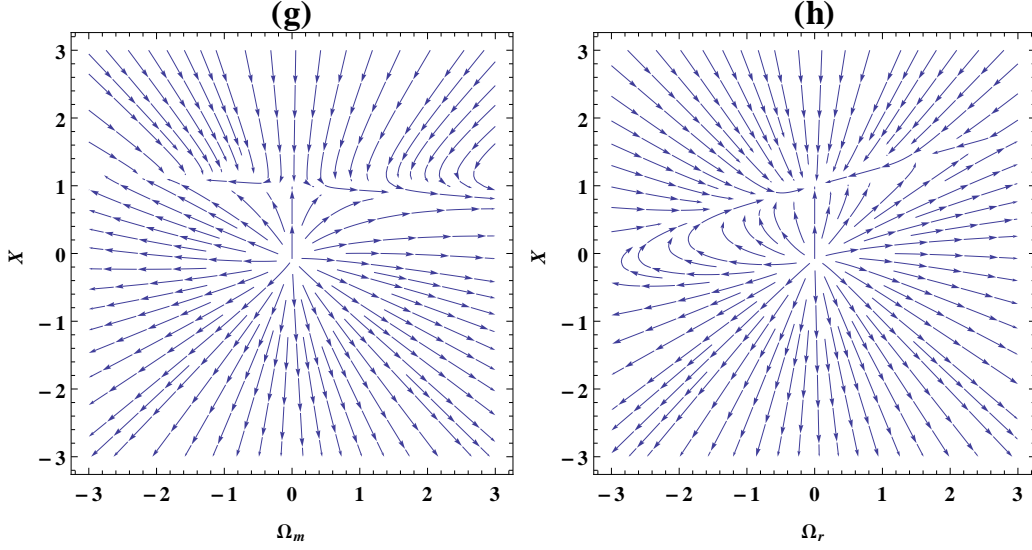


FIG. 1. The behavior of critical points in phase planes (a) $(\Omega_m, X, \Omega_r = 0)$ for $\alpha = -3.5$ (b) $(\Omega_r, X, \Omega_m = 0)$ for $\alpha = -3.5$ (c) $(\Omega_m, X, \Omega_r = 0)$ for $\alpha = .001$ (d) $(\Omega_r, X, \Omega_m = 0)$ for $\alpha = .001$ (e) $(\Omega_m, X, \Omega_r = 0)$ for $\alpha = 2$ (f) $(\Omega_r, X, \Omega_m = 0)$ for $\alpha = 2$ (g) $(\Omega_m, X, \Omega_r = 0)$ for $\alpha = 3.5$ (h) $(\Omega_r, X, \Omega_m = 0)$ for $\alpha = 3.5$. Coordinate space of P_1 and P_2 are at $(0, 1)$ and $(0, \frac{3-\alpha}{2})$ in both of the phase planes $(\Omega_r, X, \Omega_m = 0)$ and $(\Omega_m, X, \Omega_r = 0)$. P_3 is visible in the plain $(\Omega_r, X, \Omega_m = 0)$ at $(\frac{3\alpha(3-\alpha)}{2(10-3\alpha)} - \alpha + 1, \frac{3\alpha(3-\alpha)}{2(10-3\alpha)})$ and P_4 is at $(1 - \alpha, \frac{\alpha}{3})$ in the plain $(\Omega_m, X, \Omega_r = 0)$. The transition from the point $P_4(-1, \frac{2}{3})$ to $P_1(0, 1)$ in plot (e) represents the transition from matter era to acceleration phase.

TABLE III. The fixed points and physical parameters of the system (43).

fixed point	Ω_m	Ω_r	X	q
P_1	0	0	1	-1
P_2	0	0	$\frac{3-\alpha}{6}$	$\frac{\alpha-1}{4}$
P_3	0	$\frac{3\alpha(3-\alpha)}{2(10-3\alpha)} - \alpha + 1$	$\frac{\alpha(3-\alpha)}{2(10-3\alpha)}$	$1 - \frac{\alpha}{2}$
P_4	$1 - \alpha$	0	$\frac{\alpha}{3}$	$\frac{1-\alpha}{2}$

fixed point	λ_1	λ_2	λ_3	stable regime	unstable regime
P_1	$\alpha - 3$	$\alpha - 4$	$-\frac{\alpha+3}{2}$	$-3 < \alpha < 3$	—
P_2	$\frac{3}{2}(\alpha - 1)$	$\frac{3\alpha-5}{2}$	$\frac{\alpha+3}{2}$	$\alpha < -3$	$\alpha > \frac{5}{3}$
P_3	1	$\frac{5-3\alpha}{2}$	$4 - \alpha$	—	$\alpha < \frac{5}{3}$
P_4	-1	$3 - \alpha$	$\frac{3}{2}(1 - \alpha)$	$\alpha > 3$	—

By varying the action with respect to R and T , we have $\lambda = f_R$ and $\lambda' = f_T$ so:

$$S = 2\pi^2 \int dt \frac{a^3}{2k} \left\{ f(R, T) - f_R \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) \right] - f_T [T - (\rho - 3p)] + 2k\mathcal{L}_m \right\} \quad (53)$$

Integrating by parts we have for point like lagrangian of $f(R, T)$ model:

$$2k\mathcal{L} = a^3 \{ f - f_R R - f_T [T - (\rho - 3p)] \} \quad (54)$$

$$+ 6a\dot{a}^2 f_R + 6a^2 \dot{a} \dot{R} f_{RR} - 6\kappa a f_R + 6a^2 \dot{a} \dot{T} f_{RT} - 2k\omega \rho a^3$$

We want to find the consistent form of the function $f(R, T)$ by Noether symmetry for two cases;

1. $f(R, T) = h(R) + g(T)$ and we assume conservation of energy momentum tensor so we have equation (15) for $g(T)$. we can find the functionality of $h(R)$ and time behavior of the scale factor $a(t)$ by Noether symmetry.
2. $f(R, T) = R + g(T)$ but energy conservation does not exist. We find the functionality of $g(T)$ and time behavior of the scale factor $a(t)$ by Noether symmetry.

TABLE IV. Stable and unstable regimes of the parameter α for the system (43).

α regime	stable fixed point	unstable fixed point
$\alpha < -3$	P_2	P_3
$-3 < \alpha < \frac{5}{3}$	P_1	P_3
$\frac{5}{3} < \alpha < 3$	P_1	P_2
$\alpha > 3$	P_4	P_2

V.1. $f(R, T) = h(R) + g(T)$

Plugging (20) into (54) we have:

$$2k\mathcal{L} = a^3(f - f_R R - f_T T) - 6\kappa a f_R + 6a^2 \dot{a} \dot{R} f_{RR} + 6a \dot{a}^2 f_R - 6\kappa a f_R + 6a^2 \dot{a} \dot{T} f_{RT} + f_T \rho_0 (1 - 3\omega) a^{-3\omega} - 2k\omega \rho_0 a^{-3\omega}$$

The tangent space for the above lagrangian is $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}, T, \dot{T}\}$. As discussed above the generator of the Noether symmetry is:

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial T} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}} + \dot{\gamma} \frac{\partial}{\partial \dot{T}} \quad (55)$$

The Noether symmetry exist if at list one of the functions α , β and γ is different from zero. So we should solve the equation:

$$L_X \mathcal{L} = X \mathcal{L} = 0 \quad (56)$$

By replacing $\frac{d}{dt}$ in $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ as bellow:

$$\frac{d}{dt} = \frac{d}{da} \dot{a} + \frac{d}{dR} \dot{R} + \frac{d}{dT} \dot{T} \quad (57)$$

and setting the coefficient of \dot{a}^2 , \dot{R}^2 , $\dot{a}\dot{R}$, etc. equal to zero we find the following equations:

$$f_{RR} \partial_R \alpha = 0 \quad (58)$$

$$2a\alpha f_{RR} + a^2 f_{RRR} \beta + \gamma a^2 f_{RRT} + \partial_a \alpha a^2 f_{RR} \quad (59)$$

$$+ 2\partial_R \alpha f_R a + \partial_R \beta a^2 f_{RR} + a^2 f_{RT} \partial_R \gamma = 0$$

$$\alpha f_R + \beta f_{RR} a + \gamma f_{RT} a + 2f_R a \partial_a \alpha + a^2 f_{RR} \partial_a \beta + a^2 f_{RT} \partial_a \gamma = 0 \quad (60)$$

$$2\partial_T \alpha f_R a + a^2 \partial_T \beta f_{RR} + 2a\alpha f_{RT} + \beta a^2 f_{RRT} \quad (61)$$

$$+ \gamma a^2 f_{RTT} + a^2 f_{RT} \partial_T \gamma + a^2 f_{RT} \partial_a \alpha = 0$$

$$\partial_T \alpha f_{RR} = 0 \quad (62)$$

and Noether constraint:

$$3\alpha a^2 (f - f_R R - f_T T) - 6\kappa f_R \alpha - 3\alpha \omega \rho_0 [f_T (1 - 3\omega) - 2k\omega] a^{-3\omega-1} - \beta a^3 (f_{RR} R + f_{TR} T) + \beta f_{RT} \rho_0 (1 - 3\omega) a^{-3\omega} - 6\beta \kappa f_{RR} a - \gamma a^3 (f_{RT} R + f_{TT} T) - 6\kappa \gamma f_{RT} a + \gamma f_{TT} \rho_0 (1 - 3\omega) a^{-3\omega} = 0 \quad (63)$$

from (58) and (62) we have:

$$\partial_a \alpha = \partial_T \alpha = 0 \Rightarrow \alpha = \alpha(a) \quad (64)$$

by inserting the condition $f(R, T) = h(R) + g(T)$ and from (61) and (64) we have:

$$\partial_T \beta = 0 \Rightarrow \beta = \beta(a, R) \quad (65)$$

We have two equations (59) and (60) for three functions α , β and γ . So one of the functions are free and we can choose any value for it. The most simple case is $\gamma = 0$. So equations (59) and (60) become:

$$2a\alpha f_{RR} + a^2 f_{RRR} \beta + \frac{d\alpha}{da} a^2 f_{RR} + \partial_R \alpha a^2 f_{RR} = 0 \quad (66)$$

$$\alpha f_R + \beta f_{RR} a + 2f_R a \frac{d\alpha}{da} + a^2 f_{RR} \partial_a \beta = 0 \quad (67)$$

Same equations as in ref. [26] is obtained. The solutions are:

$$\alpha = c_1 a + \frac{c_2}{a}, \quad \beta = -\left(3c_1 + \frac{c_2}{a^2}\right) \frac{f_R}{f_{RR}} + \frac{c_3}{a f_{RR}} \quad (68)$$

where c_1, c_2, c_3 are constants. Plugging (68) into (63) we have:

$$(c_1 a + \frac{c_2}{a}) [3a^2 (h - R h' - \theta g_0 T^\theta) - 6\kappa h' - 3\omega \rho_0 [\theta g_0 T^\theta (1 - 3\omega) - 2k\omega] a^{-3\omega-1}] - \left[-\left(3c_1 + \frac{c_2}{a^2}\right) h' + \frac{c_3}{a} \right] (R a^3 - 6\kappa a) = 0 \quad (69)$$

from equation (50) the conserved charge of the theory is:

$$\mathcal{A} = 6c_1 a (h'' a^2 \dot{R} - h' \dot{a}) + 6c_2 (h'' a^2 \dot{R} + h' \dot{a}) + 6h' a \dot{a} c_3$$

By substituting α and β and the answer of equation (69) in the above equation we can find time behavior of the scale factor $a(t)$.

V.2. $f(R, T) = R + g(T)$

In this section we assume that energy momentum tensor is nonconserved. The point like lagrangian assuming $p = \omega \rho$ using equations (9), (10) and (54) becomes:

$$2k\mathcal{L} = a^3 (g - T g') - 6\kappa a + 6a \dot{a}^2 + [g' (1 - 3\omega) - 2k\omega] \frac{3a \dot{a}^2 - \frac{g}{2} a^3 + \kappa a}{k + (1 + \omega) g'} \quad (70)$$

where prime stands for differentiation with respect to the argument. R is absent in the lagrangian (70) so the Noether symmetry is:

$$X \mathcal{L} = \alpha \frac{\partial \mathcal{L}}{\partial a} + \gamma \frac{\partial \mathcal{L}}{\partial T} + \dot{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\gamma} \frac{\partial \mathcal{L}}{\partial \dot{T}} = 0 \quad (71)$$

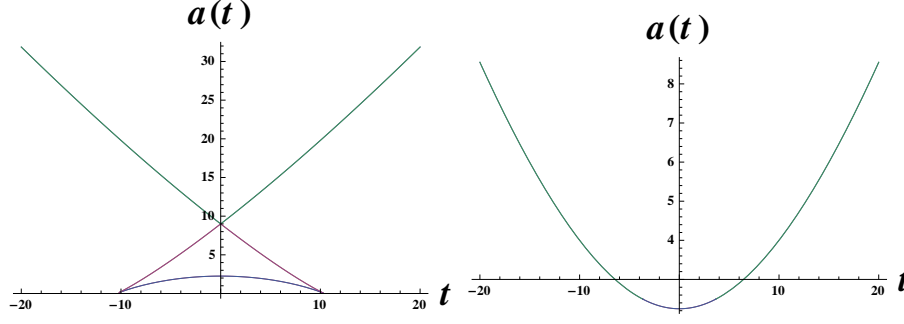


FIG. 2. The behavior of $a(t)$ against t by Noether symmetry in dS space (left figure) and AdS space (right figure)

By imposing the Noether symmetry and setting the coefficients of \dot{T}^2 , \dot{a}^2 and $\dot{T}\dot{a}$ equal to zero we find the following equations:

$$\alpha + 2a\partial_a\alpha = 0 \quad (72)$$

$$\partial_T\alpha = 0 \quad (73)$$

that we have set $\gamma = 0$ like previous section. From equations (72) and (73) α becomes:

$$\alpha = \frac{c}{\sqrt{a}} \quad (74)$$

where c is constant. Noether constraint and conserved charge are like bellow:

$$(g - Tg') - \frac{2\kappa}{a^2} + \left(-\frac{1}{2}g + \frac{\kappa}{3\alpha a^2}\right) \frac{g'(1-3\omega)-2k\omega}{k+(1+\omega)g'} = 0 \quad (75)$$

$$\sqrt{a}\dot{a} \left[2 + \frac{g'(1-3\omega)-2k\omega}{k+(1+\omega)g'} \right] = \mathcal{A} \quad (76)$$

To see the consequence of the calculations presented in this section we try to solve a simple case of above equations. Assuming $\mathcal{A} = 0$ in equation (76) $g(T)$ becomes:

$$g = g_0 T \quad g_0 = \frac{-2k(1+\omega)}{5\omega+1} \quad (77)$$

It means that in high cosmological density limit ($\omega = 1$) of the field equations by choosing $f(R, T) = R + g_0 T$ we obtain conservation of energy and Noether symmetry simultaneously. As another consequence we can find scale factor; replacing (77) in (75) the following relation is found:

$$\kappa + \frac{\kappa}{3c}\sqrt{a} - \frac{1}{2}g_0(1-3\omega)\rho a^2 = 0 \quad (78)$$

Replacing the term ρa^2 from (9) in above equation we have:

$$\kappa + \frac{\kappa}{3c}\sqrt{a} + \frac{(3\dot{a}^2 + \kappa)(1-3\omega)(\omega+1)}{5\omega+1 + (\omega-3)(\omega+1)} = 0 \quad (79)$$

The static universe is related to the flat case $\kappa = 0$ or radiation ($\omega = 1/3$). It means for radiation dominated or complete flat universe Noether symmetry leads to a static universe. Only a small curvature leads to a nonlinear differential equation above. In Fig. 2 the behavior of $a(t)$ against t is plotted for de Sitter (dS) and anti-de Sitter (AdS) spaces.

VI. CONCLUDING REMARKS

This paper dealt with the $f(R, T)$ theory of gravity. We have studied equations of motion and future singularities for a barotropic perfect fluid and a dark energy like fluid assuming conservation of energy. To keep the conservation of stress-energy tensor, the choice of $f(R, T)$ is not completely arbitrary. It was found that there is no future singularity for the barotropic fluid while some kinds of singularity possibly exist for the dark energy like fluid due to the new degrees of freedom in choosing the equation of state. We found relationships between the exponents of t in the relations of ρ , p , g and g' . We showed that it is possible to explain the expansion of the universe by an effective running coupling constant where the pressure and density produced in the equations have the same behavior as the dark energy.

Considering singularities necessitates to study a special form of the scale factor. In order to generalize the studies we turned to the method of dynamical systems. We found four fixed points which are related to radiation, matter and dark energy dominated accelerating phase of the universe. Behavior of the phase trajectories and evolution of the universe depends strongly on the value of α (the coupling constant of the interaction of the matter and radiation). There are four regimes in table IV that show different behaviors of the fixed points. In each regime evolution of the universe starts from an unstable point and ends in the stable one which is an accelerating fixed point in all cases. The preferred regime is $\frac{10}{3} < \alpha < 3$ where matter and radiation fixed points are saddle points and Ω_r has positive values. It is interesting that weak energy condition is violated for Ω_m in this regime.

Finally, the effect of the Noether symmetry on $f(R, T)$ was studied and a consistent form of this function was determined using the Noether symmetry and the conserved charge for two cases. In the first one we assumed $f(R, T) = f(R) + g(T)$ and also the conservation of energy. In the second case we defined $f(R, T) = R + g(T)$ with no requirement to conservation of energy momentum tensor. In both cases it is possible at list numerically to find the consistent form of the function $f(R, T)$ and time behavior of the scale factor a simultaneously using the Noether symmetry. In the he second case we can

also have both conservation of energy momentum tensor and Noether symmetry at the same time for $\omega = 1$. For

future research, it will be interesting to generalize this study to other types of gravitational theories.

-
- [1] A. G. Riess et al., *Astron. J.* **116** (1998) 1009; S. Perlmutter et al., *Astrophys. J.* **517** (1999) 565; P.de Bernardis et al., *Nature* **404** (2000)955; S. Perlmutter et al., *Astrophys. J.* **598** (2003) 102.
 - [2] K. Bamba, S. Nojiri, S. D. Odintsov, *JCAP* **0810** (2008) 045, arXiv:0807.2575 [hep-th],
A. d. Cruz-Dombriz, D. Saez-Gomez, *Entropy* **14** (2012) 1717-1770, arXiv:1207.2663 [gr-qc].
 - [3] K. Bamba, S. D. Odintsov, L. Sebastiani, S. Zerbini, *Eur.Phys.J. C* **67** (2010) 295-310, arXiv:0911.4390 [hep-th],
K. Bamba, R. Myrzakulov, S. Nojiri, S. D. Odintsov, *Phys.Rev. D* **85** (2012) 104036, arXiv:1202.4057 [gr-qc].
 - [4] S. Nojiri and S. D. Odintsov, *Phys. Rept.* **505** (2011) 59-144;
 - [5] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Phys. Rev. D* **70** (2001) 043528.
 - [6] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **657** (2007) 238; *Phys. Rev. D* **77** (2008) 026007; arXiv: 1008.4275 (Prog. Theor. Phys. Suppl. (to be published)); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini *Phys. Rev. D* **77** (2008) 046009; **83** (2011) 086006.
 - [7] S.Capozziello, V.F.Cardone, and A.Troisi, *J. Cosmol. Astropart. Phys.* **08** (2006) 001; *Mon. Not. R. Astron. Soc.* **375** (2007) 1423.
 - [8] A. Borowiec, W. Godlowski and M. Szydlowski, *Int. J. Geom. Methods Mod. Phys.* **4** (2007) 183;
 - [9] C. F. Martins and P. Salucci, *Mon. Not. R. Astron. Soc.* **381** (2007) 1103.
 - [10] C. G. Boehmer, T. Harko and F. S. N. Lobo, *Astropart. Phys.* **29** (2008) 386.
 - [11] C. G. Boehmer, T. Harko and F. S. N. Lobo, *Astropart. Phys.* **03** (2008) 024.
 - [12] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **84** (2011) 024020.
 - [13] F.G. Alvarenga, A. CruzDombriz, M.J.S. Houndjo, M.E. Rodrigues and D. SaezGomez, *Phys. Rev. D* **87** (2013) 103526.
 - [14] M. Sharif and M. Zubair, *JCAP* **03** (2012) 028, arXiv: 1204.0848 (gr-qc).
 - [15] T. Azizi, *Int J Theo Phys* 52:3486-3493, 2013, arXiv: 1205.6957 (gr-qc).
 - [16] F.G. Alvarenga, M.J.S. Houndjo, A.V. Monwanou and J.B. Chabi Orou, *J. Mod. Phys.* **04** (2013) 130.
 - [17] M. Jamil, D. Momeni, M. Raza and R. Myrzakulov, *Eur. Phys. J. C* **72** (2012) 1999,
M. Sharif and M. Zubair, *Astrophys. Space Sci.* **349** (2014) 529,
M.J.S. Houndjo, *Int. J. Mod.Phys. D* **21** (2012) 1250003.
 - [18] S. Shabani and M. Farhoudi, arxiv: 1407.6187v2[gr-qc]
 - [19] S.D. Odintsov and D. Saes-Gomes, *Phys. Lett. B* **725** (2013) 437,
Z. Haghani, T. Harko, F.S.N. Lobo, H.R. Sepangi and S. Shahidi, *Phys. Rev. D* **88** (2013) 044023.
 - [20] C. G. Bohmer and N. Chan, arXiv:1409.5585 [gr-qc].
 - [21] S. Carloni, A. Troisi and P. K. S. Dunsby, *Gen. Rel. Grav.* **41**, (2009) 1757.
 - [22] R. Ribeiro and J. Paramos, arXiv:1409.3046v1 [gr-qc],
C. Gao and Y. G. Shen, arXiv:1501.06960v4 [gr-qc].
 - [23] S. K. Biswasa and S. Chakraborty, arXiv:1504.02431v1 [gr-qc].
 - [24] S. Shabani and M. Farhoudi, *Phys. Rev. D* **88**, (2013) 044048, arXiv:1306.3164v4 [gr-qc].
 - [25] S. Capozziello, R. de Ritis and A. Marino, *Class. Quantum Gravity* **14** (1997) 3259.
B. Vakili, *Phys. Lett. B.* **16** (2008) 664.
B. Vakili and F. Khazaie, *Class. Quantum Grav.* **29** (2012) 035015 (Preprint arXiv:1109.3352).
S. Capozziello, S. De. Laurentis and S. D. Odintsov, *Eur. Phys. J.* **72** (2012) 2086 (Preprint arXiv:1206.4842).
 - [26] S. Capozziello and A. D. Felice, *JCAP* **08** (2008) 016, arXiv: 0804.2163 [gr-qc].
 - [27] H. Wei, X. J. GuoL. and F. Wang, *Phys. Lett. B* **707** (2012) 298.
 - [28] S. Nojiri, S. D. Odinstov and S. Tsujikawa, *Phys. Rev. D* **71** (2005) 063004.
 - [29] L. Fernandez-Jambrin, R. Lazkoz, *J. Phys.: Conf. Ser.* **229**, 012037 (2010), arXiv:1012.3051 [gr-qc]; L. Fernandez-Jambrin, *Journal of Physics: Conference Series* **314**, 012061 (2011), arXiv: 1012.3159
 - [30] S. Chakraborty, *Gen Relativ Gravit* **45** (2013) 20392052.